1. (i) Show that $\sum_{r=1}^{16}(3 + 5r + 2^r) = 131\,798$

(4)

(ii) A sequence $u_1, u_2, u_3, \ldots$ is defined by

$U_n = ar^{n-1}$

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

n^{th} term of Sum $= 2^n$
$= 2(2^{n-1})$
$ar^{n-1} = 2(2^{n-1})$
$\Rightarrow a = 2$
$r = 2$

Find the exact value of $\sum_{r=1}^{100} u_r$

(3)

i) $\sum_{r=1}^{16}(3) + \sum_{r=1}^{16}(5r) + \sum_{r=1}^{16}(2^r)$

$= \underbrace{[3+3+\ldots+3]}_{\text{16 lots of 3}} + [5(1)+5(2)+\ldots 5(16)] + [2^1 + 2^2 + \ldots 2^{16}]$

$\times 2 \therefore$ Geometric Series

$\Rightarrow 16 \times 3$

$= 48 + [5(1+2+3+\ldots+16)] + \left[\frac{(2)(2^{16}-1)}{(2)-1}\right]$

$= 48 \checkmark_1 + 680 \checkmark_1 + 131{,}070 \checkmark_1$

$= \underline{131{,}798} \checkmark_1$

Sum of Geometric Series: $\frac{a(r^n-1)}{r-1}$

(ii) A sequence $u_1, u_2, u_3, \dots$ is defined by

$$u_{n+1} = \frac{1}{u_n}, \quad u_1 = \frac{2}{3}$$

Find the exact value of $\sum_{r=1}^{100} u_r$

ii) $U_1 = \frac{2}{3}$ $\quad U_2 = \frac{1}{U_1} \Rightarrow U_2 = \frac{1}{\left(\frac{2}{3}\right)} = \frac{3}{2}$

$U_3 = \frac{1}{U_2} \Rightarrow U_3 = \frac{1}{\left(\frac{3}{2}\right)} = \frac{2}{3}$

$$\sum_{r=1}^{100} U_r = \underbrace{\frac{2}{3} + \frac{3}{2} + \frac{2}{3} + \frac{3}{2} + \dots \frac{2}{3}}_{100 \text{ terms}} + \frac{3}{2}$$

$= 50\left(\frac{2}{3}\right) + 50\left(\frac{3}{2}\right)$

$= \frac{325}{3}$

2. A sequence of numbers $a_1, a_2, a_3, \ldots$ is defined by

$$a_{n+1} = \frac{k(a_n + 2)}{a_n} \qquad n \in \mathbb{N}$$

where k is a constant.

Given that

- the sequence is a periodic sequence of order 3
- $a_1 = 2$

(a) show that

$$k^2 + k - 2 = 0$$

(3)

a) $a_{n+1} = \dfrac{k(a_n+2)}{a_n}$, What do we know ?

- $a_1 = 2$ — first/initial term
- period of order 3

$\Downarrow$ because of this we know that $\underline{a_4 = a_1}$

Since $a_1 = 2$: $a_2 = \dfrac{k(2+2)}{2} = 2k$ ①

$a_3 = \dfrac{k(2k+2)}{2k} = \dfrac{2k^2+2k}{2k} = k+1$

$a_4 = \dfrac{k(k+1+2)}{k+1} = \dfrac{k(k+3)}{k+1}$

$\Rightarrow a_4 = a_1$ ① $\Rightarrow \dfrac{k(k+3)}{k+1} = 2$

$\Rightarrow k^2 + 3k = 2k + 2 \Rightarrow \underline{k^2 + k - 2 = 0}$ as $\underline{\text{required}}$. ①

(b) For this sequence explain why $k \neq 1$

(1)

b) From part a:

$a_1 = 2$
$a_2 = 2k$
$a_3 = k+1$
$a_4 = \dfrac{k(k+3)}{k+1}$

For $k=1$, we have:

$a_1 = 2$
$a_2 = 2$
$a_3 = 2$
$a_4 = 2$

Since all the terms are the same, the sequence no longer has a period of order 3, hence $k \neq 1$ for this sequence. 1

(c) Find the value of

$$\sum_{r=1}^{80} a_r$$

(3)

c) From part a: $k^2 + k - 2 = 0$

$(k-1)(k+2) = 0$

$\Rightarrow k = 1$ and $k = -2$

this is not a valid solution (part b)

$\Rightarrow k = -2.$

$\dfrac{80}{3} = 26\frac{2}{3}$

$a_1 = 2$
$a_2 = 2k$
$a_3 = k+1$
$a_4 = \dfrac{k(k+3)}{k+1}$

$\Rightarrow$ $a_1 = 2$
$a_2 = -4$
$a_3 = -1$ 1
$a_4 = 2$

repeating terms

$\Rightarrow \sum_{r=1}^{80} a_r = 26 \times (2 - 4 - 1) + 2 - 4$ 1

$= \underline{-80}$ 1

In this question you should show all stages of your working.

Solutions relying entirely on calculator technology are not acceptable.

3. A company made a profit of £20 000 in its first year of trading, Year 1

A model for future trading predicts that the yearly profit will increase by 8% each year, so that the yearly profits will form a geometric sequence.

According to the model,

(a) show that the profit for Year 3 will be £23 328 **(1)**

(b) find the first year when the yearly profit will exceed £65 000 **(3)**

(c) find the total profit for the first 20 years of trading, giving your answer to the nearest £1000 **(2)**

(a) $u_n = ar^{n-1}$

$u_3 = 20000 \times 1.08^{(3-1)}$

$= 20000 \times 1.08^2$

$= 23\,328$ (1)

$\therefore$ Profit in Year 3 is £23 328 #

(b) $20000 \times 1.08^{n-1} > 65\,000$ (1)

$1.08^{n-1} > \frac{65\,000}{20\,000}$

$1.08^{n-1} > 3.25$

$(n-1)\ln 1.08 > \ln 3.25$

$n-1 > \frac{\ln 3.25}{\ln 1.08}$ (1)

$n > \frac{\ln 3.25}{\ln 1.08} + 1$

> 16.31

$\therefore$ Year 17 # (1)

(c) $S_{20} = \frac{20\,000(1-1.08^{20})}{1-1.08}$ (1)

$= 915\,239\ldots$

$\approx$ £ 915 000 (nearest £1000) (1)

4. In an arithmetic series

- the first term is 16
- the 21st term is 24

(a) Find the common difference of the series.
(2)

(b) Hence find the sum of the first 500 terms of the series.
(2)

a) $a_2 = a_1 + (n-1)d$ ← formula. $a_n = a_1 + (n-1)d$

$24 = 16 + (21-1) \times d$ 1

$24 = 16 + 20d$

$8 = 20d$

$0.4 = d$ 1

b) $S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$

$S_{500} = \frac{1}{2} \times 500\left[2 \times 16 + (500-1) \times 0.4\right]$ 1

$S_{500} = 250(32 + 199.6)$

$S_{500} = 250 \times 231.6$

$S_{500} = 57900$ 1

OR use $S_n = \frac{1}{2}n(a+l)$

$l = 16 + (500-1) \times 0.4$

$l = 215.6$

$S_{500} = \frac{1}{2} \times 500(16 + 215.6)$

$S_{500} = 57900$

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5. Show that

$$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{9}{28}$$

(3)

$a = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$ 1

formula for sum to infinity of geometric series.

$S_\infty = \frac{a}{1-r}$

a = first term

r = common ratio

$r = -\frac{3}{4}$

$\sum_{n=2}^{\infty}\left(\frac{3}{4}\right)^n \cos(180n)^\circ = \frac{\frac{9}{16}}{1-\left(-\frac{3}{4}\right)}$ 1

$= \frac{\frac{9}{16}}{\frac{7}{4}}$

$= \frac{9}{28}$ $\square$ 1

6. In a geometric series the common ratio is r and sum to n terms is S_n

Given

$$S_\infty = \frac{8}{7} \times S_6$$

show that $r = \pm\frac{1}{\sqrt{k}}$, where k is an integer to be found.

(4)

$S_\infty = \frac{a}{1-r}$ $\qquad$ $S_n = \frac{a(1-r^n)}{1-r}$

$$\frac{a}{1-r} = \frac{8}{7}\left(\frac{a(1-r^6)}{1-r}\right)$$

$$\frac{a}{\cancel{1-r}} = \frac{8a(1-r^6)}{7(\cancel{1-r})}$$

$$\cancel{a} = \frac{8\cancel{a}(1-r^6)}{7}$$

$$1 = \frac{8(1-r^6)}{7}$$

$$1-r^6 = \frac{7}{8}$$

$$r^6 = \frac{1}{8}$$

$$r = \frac{\sqrt[6]{1}}{\sqrt[6]{8}}$$

$$r = \frac{1}{\sqrt{2}}$$

$$r = \pm\frac{1}{\sqrt{2}}$$

$\therefore K = 2$

7. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre.
After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, **(2)**

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. **(4)**

a) Total time for first 6 km $= (6 \times 4) + (6 \times 1.05) + (6 \times 1.05^2)$ ①

minutes km

$= 36.915$ minutes $= 36$ minutes 55 seconds ①

$(60 \times 0.915 = 54.9 \approx 55)$

b) 5^{th} km : 6×1.05^1 $5-1=4$
6^{th} km : 6×1.05^2 $6-2=4$
7^{th} km : 6×1.05^3 $7-3=4$ ①

$\therefore$ it follows that the time for r^{th} km is $6 \times 1.05^{r-4}$

Geometric Series
$a = 6.3$
$r = 1.05$
$n = 16$

$$S_n = \frac{a(1-r^n)}{1-r}$$

c) total time $= 24$ minutes $+ \sum_{r=5}^{r=20} 6 \times 1.05^{r-4}$ ①

① 4km

$= 24$ minutes $+ \dfrac{6.3(1-1.05^{16})}{1-1.05}$ ①

$= 173.042$ minutes $= 173$ minutes 3 seconds ①

(60×0.042)

8. A car has six forward gears.

The fastest speed of the car

- in 1st gear is 28 km h^{-1}
- in 6th gear is 115 km h^{-1}

Given that the fastest speed of the car in successive gears is modelled by an **arithmetic sequence**,

(a) find the fastest speed of the car in 3rd gear.

(3)

a) Arithmetic Sequence : $a_n = a + (n-1)d$

a_n = n'th term
a = first / initial term (28 kmh^{-1})
d = Common difference between terms.

$a = 28$, $a_6 = 115$

$\Rightarrow a_6 = 115 = 28 + (6-1) \cdot d$

$\Rightarrow 5d = 115 - 28 \quad \Rightarrow d = \dfrac{115-28}{5} = \underline{17.4}$ 1

$\Rightarrow a_3 = 28 + (3-1)17.4$ 1

$\Rightarrow a_3 = \underline{62.8}$ kmh^{-1} is the fastest speed of the car in 3rd gear. 1

Given that the fastest speed of the car in successive gears is modelled by a **geometric sequence**,

(b) find the fastest speed of the car in 5th gear.

(3)

b) Geometric Sequence : $a_n = ar^{n-1}$

a_n = n'th term
a = first / initial term
r = Common ratio between terms

$a_6 = 115$ kmh^{-1} and $a = 28$ kmh^{-1}

$\Rightarrow a_6 = 115 = 28 \cdot r^5$

$\Rightarrow r^5 = \dfrac{115}{28} \quad \Rightarrow r = \left(\dfrac{115}{28}\right)^{1/5} = 1.3265\ldots$ 1

$\Rightarrow a_5 = 28 \cdot (1.3265\ldots)^4 = 86.6941\ldots$ 1 $\Rightarrow a_5 = \underline{86.7}$ kmh^{-1} is the fastest speed of the car in 5th gear. 1

9. A geometric series has common ratio r and first term a.

Given $r \neq 1$ and $a \neq 0$

(a) prove that

$$S_n = \frac{a(1-r^n)}{1-r}$$

(4)

$S_n = a + ar + ar^2 + \dots + ar^{n-1}$ $\times r$ $\times r$ 1

$\Rightarrow r \cdot S_n = ar + ar^2 + ar^3 + \dots + ar^n$ 1

$\Rightarrow S_n - r \cdot S_n = a + ar + ar^2 + \dots + ar^{n-1} - ar - ar^2 - \dots - ar^n$

$\Rightarrow S_n - r \cdot S_n = a - ar^n$ 1 (we now want to rearrange and manipulate this to get the required answer / proof)

$\Rightarrow S_n(1-r) = a(1-r^n)$ $\Rightarrow$ $S_n = \dfrac{a(1-r^n)}{1-r}$ as required. 1

$\div (1-r)$

Given also that S_{10} is four times S_5

(b) find the exact value of r.

(4)

Recall that : $S_n = \dfrac{a(1-r^n)}{1-r}$

$\Rightarrow S_{10} = 4 \times S_5$

$\Rightarrow \dfrac{a(1-r^{10})}{1-r} = \dfrac{4a(1-r^5)}{1-r}$ $\div a$ $\times 1-r$ 1

$\Rightarrow 1 - r^{10} = 4(1-r^5)$ 1 $\Rightarrow 1 - r^{10} = 4 - 4r^5$

$\Rightarrow r^{10} - 4r^5 + 3 = 0$ then let $x = r^5$ and $x^2 = r^{10}$

$\Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x-3)(x-1) = 0$

$\Rightarrow (r^5-3)(r^5-1) = 0$ $\rightarrow$ $r^5 = 1 \Rightarrow r = 1$ (but this solution isn't valid since $r \neq 1$).

$\Rightarrow r^5 = 3$ 1

$\Rightarrow r = \sqrt[5]{3}$

$\Rightarrow$ The exact value of r is $r = \sqrt[5]{3}$ 1